Exercise 63

If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$.

Solution

Take the derivative of the function.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^a(1-x)^b] \\ &= \left[\frac{d}{dx}(x^a)\right](1-x)^b + x^a\left[\frac{d}{dx}(1-x)^b\right] \\ &= (ax^{a-1})(1-x)^b + x^a\left[b(1-x)^{b-1}\cdot\frac{d}{dx}(1-x)\right] \\ &= ax^{a-1}(1-x)^b + x^a[b(1-x)^{b-1}\cdot(-1)] \\ &= ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} \\ &= \frac{ax^a(1-x)^b}{x} - \frac{bx^a(1-x)^b}{1-x} \\ &= \left(\frac{a}{x} - \frac{b}{1-x}\right)x^a(1-x)^b \\ &= \left[\frac{a(1-x)-bx}{x(1-x)}\right]x^a(1-x)^b \\ &= \frac{[a(1-x)-bx]x^a(1-x)^b}{x(1-x)} \end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve each equation for x.

$$\begin{aligned} x^a(1-x)^b[a(1-x)-bx] &= 0 & x(1-x) &= 0 \\ x^a = 0 \quad\text{or}\quad (1-x)^b = 0 \quad\text{or}\quad a(1-x)-bx &= 0 & x = 0 \quad\text{or}\quad 1-x &= 0 \\ x = 0 \quad\text{or}\quad 1-x = 0 \quad\text{or}\quad a-ax-bx &= 0 & x = 0 \quad\text{or}\quad x &= 1 \\ x = 0 \quad\text{or}\quad x = 1 \quad\text{or}\quad a-(a+b)x &= 0 & x = 0 \quad\text{or}\quad x &= 1 \\ x = 0 \quad\text{or}\quad x = 1 \quad\text{or}\quad x &= \frac{a}{a+b} & x = 0 \quad\text{or}\quad x &= 1 \end{aligned}$$

$x = 0$ and $x = 1$ are within $[0, 1]$, so evaluate f at these values. Additionally, since a and b are both positive, $a/(a+b)$ is also within $[0,1]$.

$$f(0) = 0^a(1-0)^b = 0 \qquad \text{(absolute minimum)}$$

$$f(1) = 1^a(1-1)^b = 0 \qquad \text{(absolute minimum)}$$

$$\begin{aligned} f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a \left[1 - \left(\frac{a}{a+b}\right)\right]^b \\ &= \left(\frac{a}{a+b}\right)^a \left[\frac{(a+b)-a}{a+b}\right]^b \\ &= \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b \\ &= \frac{a^a b^b}{(a+b)^{a+b}} \qquad \text{(absolute maximum)} \end{aligned}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $0 \le x \le 1$.